4 THE MATHEMATICAL MODEL

In this part of the thesis I am going to formulate the mathematical problem that shall be optimized. It could be done in a strict mathematical way, but I think it is preferable to start with one easy retrofit problem and then show how this could be optimized. Having solved this first one, I will make the model more complex by adding a new retrofit and show how this new problem can be solved in order to find the minimum LCC. The disposition of the thesis then in a way follows my own process of studying the problem.

It is obvious that the first thing to do must be to formulate the engineering optimization problem. As is said in (7) this is the key to the success of optimization and is for bigger problems to a large degree an art.

In my case I will try to find the lowest possible LCC for an existing building. The house already has an LCC and by making retrofit measures to this it might be possible to make the new LCC lower. I have chosen to build up the model with the use of a numerical example and I also think that this makes the thesis a little easier to read.

This numerical example are all fictional, I have constructed it in order to evaluate the optimization process. However, the example must have a connection to the reality, and all the input data picked out for the house have been chosen from literature about real houses.

4.1 THE EXISTING HOUSE, NUMERICAL EXAMPLE

Input data:

Geometry

Number of apartments	20
Number of storeys	2

Each apartment has the area	100 m^2
Building area (10 apartments x x 100 m^2)	1 000 m^2
Total net dwelling area (20 apartments x 100 m^2)	2 000 m^2
The length of the house	50 m
The breadth of the house	20 m
The hight of the external walls	6 m
Area of the floor and attic floor	1 000 m^2
Area of the external wall (windows excluded)	700 m^2
Number of windows to the north	30
Number of windows to the east	12
Number of windows to the south	30
Number of windows to the west	12
Area of one window	1.7 m^2

Existing U-values

Attic floor	0.8 (W/m^2 K)
External wall	1.05 (W/m^2 K)
Floor	0.56 (W/m^2 K)
Windows north	+2.6 (W/m^2 K)
Windows east	+0.8 (W/m^2 K)
Windows south	-1.2 (W/m^2 K)
Windows west	+0.8 (W/m^2 K)
Windows darkness	4 (W/m^2 K)

Remaining life for the envelope

Attic floor	50 years
Floor	20 years
External wall	10 years
Windows	10 years

Ventilation system

Type	Natural ventilation
Ventilation flow	0.8 renewals/hour
Remaining life	50 years

Heating system

Boiler	oilheated
Efficiency	0.7
Power	170 kW
Remaining life	5 years

Tapwater flow

Energy use for tapwater heating	80 000 kWh/year

Many of the figures above come from a lot of different references. I will deal with these further down in the thesis under the suiting headline for windows, attic floors, etc.

4.1.1 The attic floor, thermal performance

The existing attic floor in my fictional building is constructed as is shown in Figure 9.

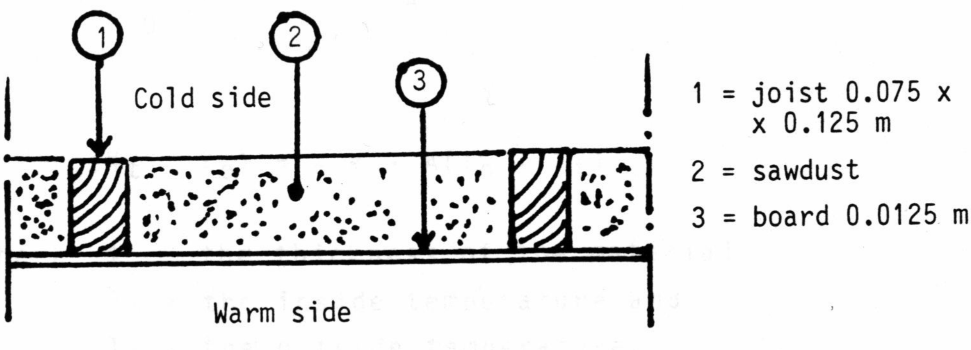


Figure 9. The existing attic floor.

In (26 p 2) the Fourier´s law of conduction is presented as:

$$q = - k \times A \frac{\delta T}{\delta x} \qquad \text{(F4)}$$

where q = the heat transfer rate

A = the area

$\delta T/\delta x$ = the temperature gradient in the direction of the heat flow

k = thermal conductivity of the material.

This equation defines the thermal conductivity constant and k has the units of W/m x K, when the heat flow is expressed in watts. In (68 p 32 -) the basic therories are explained more elaborately than is possible in this thesis.

Integrating this equation I will have:

$$\int q = \int - k \cdot A \cdot \frac{\delta T}{\delta x}$$

$$q \int_0^t dx = - k \cdot A \int_{T_i}^{T_o} dT$$

$$q(X_t - X_o) = - k \cdot A(T_o - T_i)$$

where t = the thickness of the material

T_i = the inside temperature and

T_o = the outside temperature.

X_t = t and

X_o = 0

thus

$$q = - k \cdot A \cdot \frac{T_o - T_i}{t} \qquad \text{(F5)}$$

Usually the value for k/t is called the U-value with the unit W/m^2, K and the formula:

$$q = U \cdot A(T_i - T_o)$$

is given in (22 p 3) and called the "wall heat transfer equation".

This expression only considers heat transferred in one direction and under steady-state circumstances. Sometimes the errors from this can be considerable, e g in corners and studies of special construction elements, but as an average measure for the heat transferred through the attic floor these effects can be neglected.

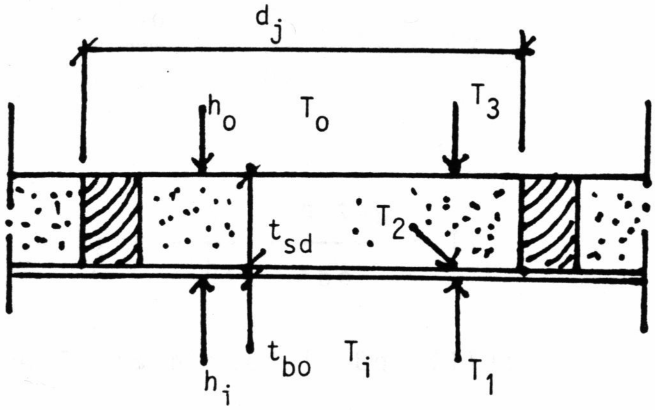
The heat has also to be transferred from and to the ambient air. This is done by convection and the heat transferred can be calculated with Newtons law of cooling (26 p 12).

$$q = h \cdot A(T_{af} - T_{\infty}) \qquad \text{(F6)}$$

where T_{af} = the temperature at the attic floor
T_{∞} = the temperature in the ambient air
h = convection heat transfer coefficient (unit (W/m^2 K)

As is obvious from Chapter (7 in 26) it is very hard to calculate adequate values for h. It is therefore necessary to use empirical values for h, and in the Swedish building code (5) it is proposed which values to be chosen. There are also other methods that can be used and in (27 p 78) h is calculated by using a formula considering the influence of the temperature. (In this book (27), written in Swedish, there are also several references to English literature.

Looking at the Figure 10 for an attic floor the heat transferred through it can be written (for a unit area):



T_o = outside temperature
T_i = inside temperature
d_j = distance between the joists
t_{bo} = thickness of the boarding
t_{sd} = thickness of the sawdust
h_o = convection coefficient at the outside
h_i = convection coefficient at the inside

Figure 10. The existing attic floor. Thermal parameters.

From (F6) and (F7) and the area = 1.

$q_1 = h_i \cdot 1(T_i - T_1)$	heat through the convection layer	I
$q_2 = k_b \cdot 1(T_1 - T_2)/t_{bo}$	heat conducted through the board	II
$q_3 = k_{sd} \cdot 1(T_2 - T_3)/t_{sd}$	heat conducted through the sawdust	III
$q_4 = h_o \cdot 1(T_3 - T_o)$	heat convected through the boundary layer	IV
$q_1 = q_2 = q_3 = q_4 = q$		V

From Eq I and V:

$$T_1 = -\frac{q}{h_i} + T_i$$

T_1 placed in Eq II gives me:

$$q = \frac{k_{bo}\left[(\frac{q}{h_i} + T_i) - T_2\right]}{t_{bo}}$$

$$\frac{q \cdot t_{bo}}{k_{bo}} = - \frac{q}{h_i} + T_i - T_2$$

$$T_2 = - \frac{q}{h_i} - \frac{q \cdot t_{bo}}{k_{bo}} + T_i$$

Now T_2 is placed in Eq (III):

$$q = \frac{k_{sd}\left[\left(- \frac{q}{h_i} - \frac{q \cdot t_{bo}}{k_{bo}} + T_i \right) - T_3 \right]}{t_{sd}}$$

$$\frac{q \cdot t_{sd}}{k_{sd}} = - \frac{q}{h_i} - \frac{q \cdot t_{bo}}{k_{bo}} + T_i - T_3$$

which gives an expression for T_3:

$$T_3 = - \frac{q}{h_i} - \frac{q \cdot t_{bo}}{k_{bo}} - \frac{q \cdot t_{sd}}{k_{sd}} + T_i$$

This can be placed in Eq (IV):

$$q = h_o\left[\left(- \frac{q}{h_i} - \frac{q \cdot t_{bo}}{k_{bo}} - \frac{q \cdot t_{sd}}{k_{sd}} + T_i \right) - T_o \right]$$

$$\frac{q}{h_o} = - \frac{q}{h_i} - \frac{q \cdot t_{bo}}{k_{bo}} - \frac{q \cdot t_{sd}}{k_{sd}} + T_i - T_o$$

$$T_i - T_o = q \left[\frac{1}{h_i} + \frac{t_{bo}}{k_{bo}} + \frac{t_{sd}}{k_{sd}} + \frac{1}{h_o} \right]$$

Chosing the same form as before:

$$q = \frac{T_i - T_o}{\frac{1}{h_i} + \frac{t_b}{k_b} + \frac{t_{sd}}{k_{sd}} + \frac{1}{h_o}} \qquad (F7)$$

and we see that the U-value in $q = U \cdot A(T_i - T_o)$ corresponds to

the expression $\frac{1}{\frac{1}{h_i} + \frac{t_{bo}}{k_{bo}} + \frac{t_{sd}}{k_{sd}} + \frac{1}{h_o}}$

Often the R-value or "the thermal resistance" is used instead of the U-value. The connection between them is:

$$U = \frac{1}{R}$$

and thus

$$R = \frac{1}{h_i} + \frac{t_b}{k_b} + \frac{t_{sd}}{k_{sd}} + \frac{1}{h_o}$$

In (22 p 4) another formula is mentioned viz

$$R = \sum_i R_i$$

and in this expression "R_i" is the thermal resistance for building part i and the sum of all those will give the total R.

Using all these above formulas makes it possible to calculate the U-value for the existing attic floor in Figure 9. Because of the uniform thermal conductivity for wood and sawdust, 0.14 $W/m^2\ ^oC$, no differentiation between the materials has to be made. $(1/h_i + 1/h_o)$ can be assumed to 0.26 $m^2\ ^oC/W$. All these values are picked from (28 p 4:5, 11 and 12).

This will give:

$$U_{af} = \frac{1}{0.26 + \dfrac{0.125+0.0125}{0.14}} = 0.80\ W/m^2\ K$$

where U_{af} = The thermal transmittance for the attic floor.

References (5) and (28) are unfortunately written in Swedish. In (26 p 538 and p 13) "k" for yellow pine is given to 0.147 $W/m^2\ ^oC$ and h to 4.5 $W/m^2\ ^oC$, but (26) is not a reference for building material constants so (28) is used instead. In (5 p 209) $(1/h_i + 1/h_o)$ are proposed to 0.25 $m^2\ ^oC/W$, but this will not have any considerable influence on U_{af}.

To calculate the maximum heat flow that is transferred through the attic floor in order to find the heat demand for the house during the worst climate conditions, it is necessary to find out the lowest outside temperature common for the site. In (5 p 237) this temperature is called LUT (Swedish Abbreviation for Lowest Outside Temperature). For houses of stone, with a high heat capacity, it is possible to calculate with LUT 5. In our case the house is made from wood and LUT 1 (5 p 235) is proposed. For Malmö, LUT 1 = = -16 ^oC. In a subsequent Chapter (no 7) this is discussed more elaborate, because some retrofit measures have a very significant influence on the so-called time constant for the house and thus it is possible to calculate with a higher outside temperature.

I have now found the thermal performance of the existing attic floor. In order to lower the heat transferred it is possible to put more insulation on the top of the floor depicted in Figure 11.

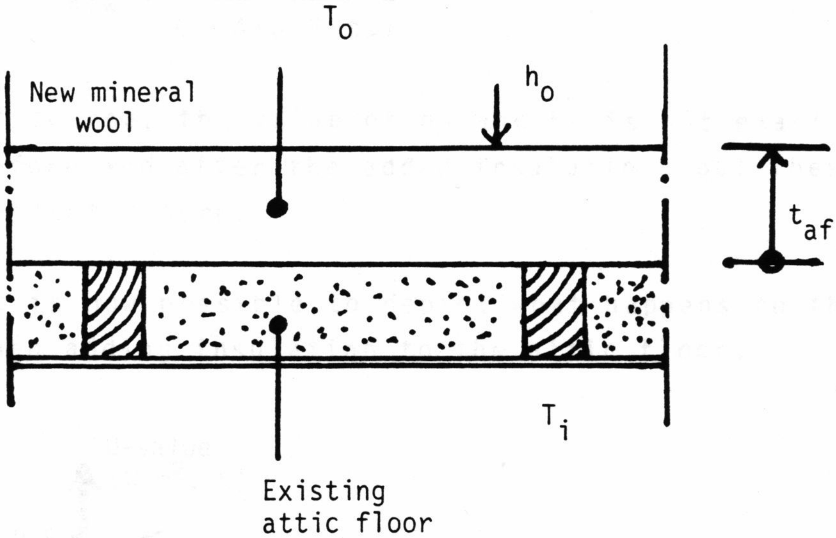


Figure 11. The retrofitted attic floor.

The new insulation of thickness t_{af} gives a lower U-value to the attic floor. This can be calculated in the following way.

From Eq (7) it could be found that adding insulation to the existing attic floor will change the equation to (mw = index for mineral wool):

$$q_{new} = \frac{(T_i - T_o) \cdot A}{\frac{1}{h_i} + \frac{t_{bo}}{k_{bo}} + \frac{t_{sd}}{k_{sd}} + \frac{t_{af}}{k_{mw}} + \frac{1}{h_o}}$$

The new U-value can be written as

$$U_{new} = \frac{1}{\frac{1}{U_{exist}} + \frac{t_{af}}{k_{mw}}} = \frac{k_{mw} \times U_{exist}}{k_{mw} + U_{exist} \times t_{af}}$$

Putting in the values for U_{exist} = 0.80 W/m^2 ^oC and k_{mw} = 0.04 W/m^2 K (28 p 4:11) we find that the new U-value will become

$$U_{new} = \frac{0.04 \cdot 0.8}{0.04 + 0.8 \cdot t_{af}} \qquad (F8)$$

Of course, the value of h_o and h_i is not exactly the same before and after the added insulation, but these effects are neglected here.

It is now possible to depict what happens to the U-value when adding insulation to the attic floor.

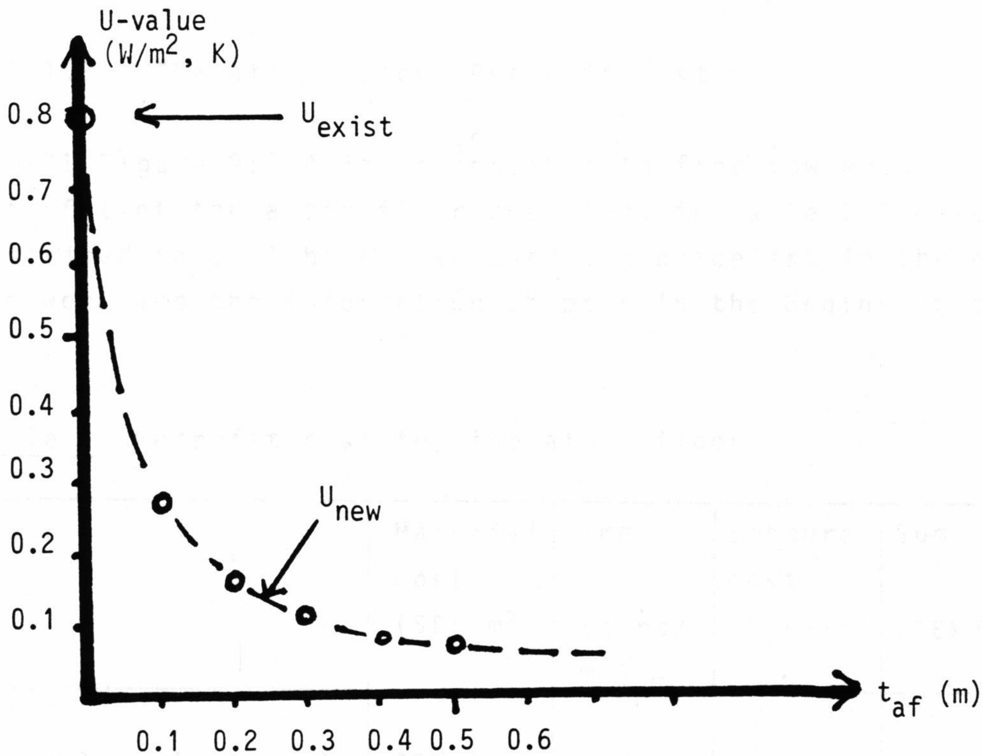


Figure 12. U-value for the retrofitted attic floor.

It is obvious that the effectiveness is highest for the first decimeters of insulation. Insulating the attic floor with 30 cm mineral wool makes the U-value result in 0.11 W/m^2 K. Adding 10 cm extra to this only makes the

U-value 0.02 W/m^2 K better.

The heat transferred through the new attic floor will be, with the actual figures, from (F8):

$$q = \frac{0.04 \times 0.08 \times 1000 \times (20-(-16))}{0.04+0.8 \times t_{af}} = \frac{1152}{0.04+0.8 \times t_{af}}$$

t_{af} = 0 gives the heat transferred for the existing attic floor = 28.8 kW. 30 cm new insulation (t = 0.3) diminutes this demand to 4.1 kW!

4.1.1.2 The attic floor. Retrofit cost

In (31 Figure 9:37) it is possible to find how much a retrofit of the attic floor costs and in Table I I have described this. I have also used the pricelist in the end of the work and the information chapter in the beginning of (31).

Table I Retrofit cost for the attic floor

	Material cost (SEK/m^2)	Time (hours/ /m^2)	Labourer cost (SEK/m^2)	Sum (SEK/m^2)
Mineral wool Type 1	270xt_{af}	0.07	4.06	4.06+270xt_{af}
Mineral wool Type 2	230xt_{af}	0.08	4.64	4.64+230xt_{af}
Plastic sheet	5.80	0.04	2.32	8.12
Gypsum board 0.013 m	17.20	0.28	16.24	33.44
Sum wages			27.26	
Indirect cost 181 % of the wage cost				49.34

Total cost 99.60 + 250xt_{af}

Taxes 12.85 % 12.81 + 32 x t_{af}

Total cost = 112.41 + 282 x t_{af}

Mineral wool of two different types are sold for attic insulation purposes and it is possible to vary the thickness of the two layers. Thus I have chosen to take the mean of the two variable costs for my calculations here. The cost for wages is 58 SEK/hour. I have also tested my formula on some cases in (31) and found that the costs calculated my way are a little too low. I have therefore approximated the expression to:

125 + 300 x t_{af} SEK/m^2 or for the actual attic floor area:

$$RC_{af} = 125\ 000 + 300\ 000 \times t_{af} \text{ SEK} \qquad (F9)$$

Going back to the thermal performance it shall be noted that the influence of the plastic shield and the gypsum board have been neglected. This is convenient because of the easier calculation process. The gypsum board etc are not considered as an energy retrofit measure and the errors due to this are very small. (R for 0.013 m gypsum board is about 0.06 m^2 K/W, (28 p 4:9). However, it is possible to include the gypsum board in (F8) by use of a slightly different k_{mn}. This procedure is more elaborate described in the next chapter considering the external wall. Looking at our input data we see that the remaining life for the attic floor is more than 50 years. This means that we don´t have any "non-energy retrofit" to think of when calculating the LCC.

4.1.1.3 The attic floor. Energy cost

In (F8) I have described a way to calculate the resulting U-value of the retrofitted attic floor. However, it is more interesting to find the cost for the energy "consumed" by the attic during its life-cycle, and I therefore have to find the amount of energy consumed. As mentioned before the inside temperature is considered to be constant, 20 °C. The outside temperature varies with the time. In this calculation of the LCC I assume that the energy consumed can be calculated by the use of the mean month temperature for the site Malmö in Sweden. In (28) these values are published.

Table II Mean month temperatures (^{o}C) and degree hours in Malmö, Sweden

	Jan	Febr	March	April	May	June	
Mean outside temperature	-0.5	-0.7	1.4	6.0	11.0	15.0	
Difference from inside temperature	20.5	20.7	18.6	14.0	9.0	5.0	
Number of hours	744	678	744	720	744	720	
Degree hours	15252	14035	13838	10080	6696	3600	
	July	**Aug**	**Sept**	**Oct**	**Nov**	**Dec**	**SUM**
Mean outside temperature	17.2	16.7	13.5	8.9	4.9	2.0	
Difference from inside temperature	2.8	3.3	6.5	11.1	15.1	18.0	
Number of hours	744	744	720	744	720	744	
Degree hours	2083	2455	4680	8258	10872	13392	105241

(The degree hour line is calculated as 20.5 x 744 + 20.7 x x 678 + 18.6 x 744 + ... = 105 241.)

Further down I will discuss the use of other definitions of the degree hour concept and the use of other inside temperatures. (See also (82) and (84) where other means is described to calculate the number of degree hours.

The energy consumption now can be calculated as

$$E_{af} = A_{af} \times U_{af} \times D$$

where A_{af} = Area of the attic floor
U_{af} = U-value for attic floor
D = Number of degree hours and
E_{af} = The energy consumption for the attic floor

In our numerical example

$$E_{af} = 1\,000 \times \frac{0.04 \times 0.8}{0.04 + 0.8 \times t_{af}} \times 105\,241 \times 3\,600 =$$

$$= \frac{12\,123}{0.04 + 0.8 \times t_{af}} \text{ MJ}$$

With no extra insulation at all, $t_{af} = 0$, the consumption will be about 303 GJ, or 84 000 kWh, each year.

Adding 0.3 m insulation will diminute this to 43 GJ or 12 000 kWh.

The energy price/kJ depends on the type of heating system, but this part of the problem has not been treated yet in this theses. Therefore, I will assume that the price is 0.0833 SEK/MJ or 0.30 SEK/kWh in this first example. Then the annual cost for energy is :

$$\text{Annual energy cost} = \frac{1010.25}{0.04 + 0.8 \times t_{af}}$$

In order to find the energy cost for the attic floor for the whole life-cycle I have to use the expression (F3) that transforms the annual recurring events to the base year. Using r = 0.05 and b = 50 the present value for the energy cost will become:

$$EC_{af} = 18.26 \times \frac{1010.25}{0.04 + 0.8 \times t_{af}} = \frac{18438.5}{0.04 + 0.8 \times t_{af}} \qquad \text{(F10)}$$

4.1.1.4 The attic floor, LCC

Adding the energy cost (F10) to the insulation cost (F9) gives the total LCC.

$$LCC_{af} = 125\ 000 + 300\ 000 \times t_{af} + \frac{18438.5}{0.04+0.8 \times t_{af}} \quad (F11)$$

This expression has been depicted in Figure 13.

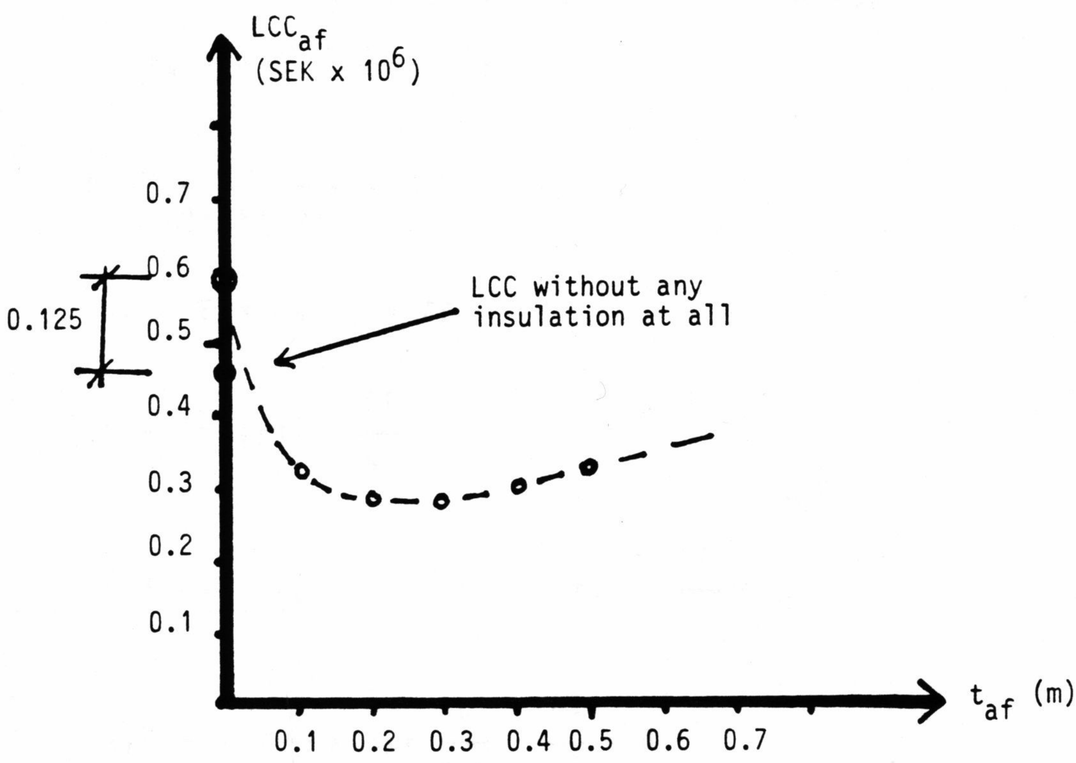


Figure 13. The LCC function for the retrofitted attic floor.

In Figure 13 we see that the existent LCC for the attic floor is lower than the LCC for the extra insulated one, when t_{af} is small. This first LCC only consists of the energy cost i e (F10), where $t_{af} = 0$. However, we see that adding more insulation will give us a much lower LCC, but when the insulation becomes more than about 0.3 m the LCC rises again.

4.1.1.5 The attic floor. Optimization technique

The minimum of the expression (F11) can easily be found by examining, where the derivative f1 equals 0, see (33) p 4. For convenience I shall write (F11) as:

$$f(t_{af}) = A + B \times t_{af} + \frac{C}{D+F \times t_{af}}$$

$$f^1(t_{af}) = B + \frac{(D+E \times t_{af}) \times 0 - C \times E}{(D+E \times t_{af})^2} = B - \frac{CE}{(D+E \times t_{af})^2}$$

$f^1(t_{af}) = 0$ gives us:

$$B - \frac{CE}{(D+E \times t_{af})^2} = 0$$

$$B(D + E \times t_{af})^2 = CE$$

$$(D + E \times t_{af})^2 = \frac{CE}{B}$$

$$D^2 + E^2 \times t_{af}^2 + 2D \times E \times t_{af} - \frac{CE}{B} = 0$$

$$t_{af}^2 + 2 \times \frac{D}{E} \times t_{af} - \frac{C}{B \times E} - \frac{D^2}{E^2} = 0$$

$$t_{af} = -\frac{1}{2} \times 2 \times \frac{D}{E} \pm \sqrt{\frac{D^2}{E^2} + \frac{C}{B \times E} - \frac{D^2}{E^2}}$$

$$t_{af} = -\frac{D}{E} + \sqrt{\frac{C}{B \times E}} \qquad \text{(F12)}$$

or with the figures from the numerical example:

$$t_{af} = -\frac{0.04}{0.8} + \sqrt{\frac{18438.5}{300000 \times 0.8}} = -0.05 + 0.28 = 0.23$$

Chosing the thickness of extra insulation to 0.23 m, thus will give us the lowest possible LCC for the attic floor. This can be calculated from (F11) to:

$$\min LCC_{af} = 125\ 000 + 300\ 000 \times 0.23 + \frac{18438.5}{0.04 + 0.8 \times 0.23} =$$

$$= 276\ 314 \text{ SEK}$$

In (7) and other literature about optimization techniques (F11) is called the objective function that shall be minimized. There are also two constraints that have to be considered viz, the LCC shall not be greater than the LCC for the existent attic floor, and the variable t_{af} must be greater than or equal to 0.

Thus our problem can be expressed like:

$$\text{Minimize: } Y_1 \times \left(125\ 000 + 300\ 000 \times t_{af} + \frac{18438.5}{0.04 + 0.8 \times t_{af}}\right) +$$

$$+ Y_2 \times \frac{18438.5}{0.04 + 0.0}$$

Subject to: $t_{af} > 0$

$Y_1, Y_2 = 1$ or 0 (integers)

$Y_1 + y_2 = 1$

If $Y_2 = 1$ then $t_{af} = 0$

In (7) p 22, (32) p 5 or in (33) p 82 the therminology is explained more elaborately. We see that when $Y_1 = 1$ the solution to the problem is described by the procedure showed above and if $Y_2 = 1$ the solution is elementary, $t_{af} = 0$, or no extra insulation shall be made.

In our numerical example the solution to the problem is $t_{af} = 0.23$, $Y_1 = 1$, $Y_2 = 0$ and these values gives the objective function the value 246 314.

4.1.2.1 The external wall. Thermal performance

Figure 14 shows the construction of the existing wall.

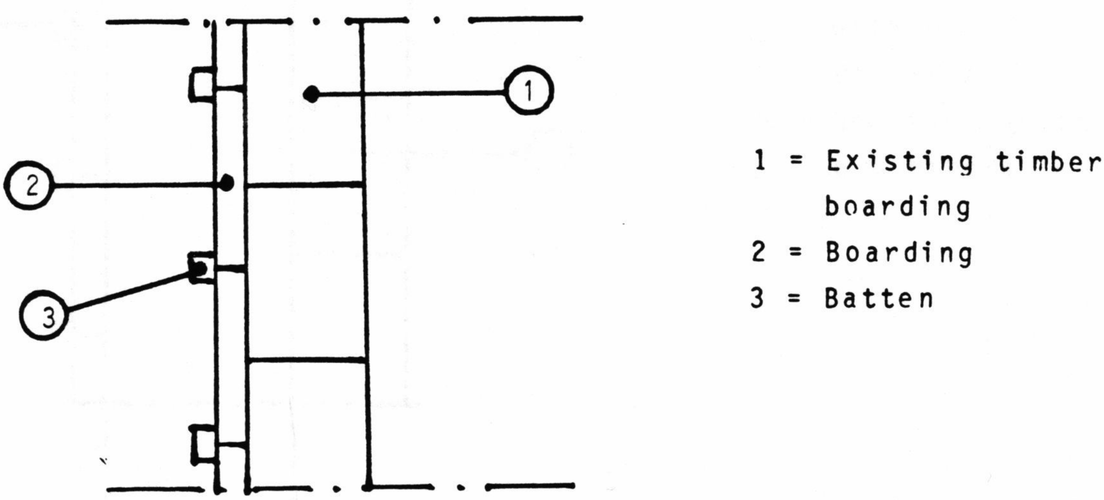


Figure 14. The existing external wall.

In exactly the same way as for the attic floor we can calculate a U-value for the existing external wall. This can be calculated as: